FLOER HOMOLOGY OF LAGRANGIAN SUBMANIFOLDS

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ABSTRACT. This paper is a survey of Floer theory, which the author has been studying jointly with Y.-G. Oh, H. Ohta, K. Ono. A general idea of the construction is outlined. We also discuss its relation to (homological) mirror symmetry. Especially we describe various conjectures on (homological) mirror symmetry and various partial results towards those conjectures.

1. INTRODUCTION

This article is a survey of Lagrangian Floer theory, which the author has been studying jointly with Y.-G. Oh, H. Ohta, K. Ono. A major part of our study was published as [28].

Floer homology is invented by A. Floer in 1980's. There are two areas where Floer homology appears. One is symplectic geometry and the other is topology of 3-4 dimensional manifolds (more specifically the gauge theory). In each of the two areas, there are several different Floer type theories. In symplectic geometry, there are Floer homology of periodic Hamiltonian system ([18]) and Floer homology of Lagrangian submanifolds. Moreover there are two different kinds of Floer theories of contact manifolds ([15, 78]). In gauge theory, there are three kinds of Floer homologies: one based on Yang-Mills theory ([17]), one based on Seiberg-Witten theory ([55, 51]), and Heegard Floer homology ([63]). Those three are closely related to each other.

All the Floer type theories have common feature that they define some kinds of homology theory of $\infty/2$ degree in ∞ dimensional space, based on Morse theory.

There are many interesting topics to discuss on the general feature of Floer type theories. I however do not discuss them in this article and concentrate on the points which are important in Lagrangian Floer theory.

2. FLOER HOMOLOGY OF LAGRANGIAN SUBMANIFOLDS

A *symplectic manifold* is a pair (X, ω) of $(2n)$-dimensional manifold X and its closed two form ω such that ω^n is nowhere 0 in X. A *Lagrangian submanifold* L of (X, ω) is an n-dimensional submanifold such that the restriction of ω to L is 0.

Typical examples of X are cotangenet bundle T^*M of a manifold M, and a Kähler manifold. Typical examples of L are zero section of T^*M, and the set of real points of a projective algebraic variety X defined over $\mathbb{R}$.

The Floer homology $HF(L_1, L_2)$ of Lagrangian submanifolds is an invariant of a pair (L_1, L_2) of Lagrangian submanifolds in a symplectic manifold X. This is actually the first one [16] among various Floer type theories that was studied by A. Floer. However there are several difficulties to establish it in the general situation and so it takes much time for such theory to be established.

The ideal properties that Floer homology of Lagrangian submanifolds are expected to enjoy can be summarized as follows.

(i) We can associate a module $HF(L_1, L_2)$ to a symplectic manifold (X, ω) and a pair (L_1, L_2) of Lagrangian submanifolds of it. $HF(L_1, L_2)$ is called the *Floer cohomology*.

(ii) A pair of Hamitonian diffeomorphims[1] $\varphi_i : X \to X$ induces an isomorphism $HF(L_1, L_2) \cong HF(\varphi_1(L_1), \varphi_2(L_2))$.

(iii) If $L_1 = L_2 = L$ then $HF(L, L) \cong \bigoplus_{i=0}^{n} H^i(L)$. Here the right hand side is the singular homology of L.

(iv) If L_1 intersects transversaly to L_2, then $HF(L_1, L_2)$ is generated by at most $\#(L_1 \cap L_2)$ elements. Here $\#(L_1 \cap L_2)$ is the order of the intersection $L_1 \cap L_2$.

If we assume (i)(ii)(iii)(iv) above, then for any pair (φ_1, φ_2) of diffeomorphisms we have

$$(2.1) \qquad \#(\varphi_1(L) \cap \varphi_2(L)) \geq \sum_{i=0}^{n} \operatorname{rank} H^i(L)$$

provided that $\varphi_1(L)$ is transversal to $\varphi_2(L)$. This is a 'Lagrangian version of Arnold's conjecture' [2], that implies a similar Arnold's conjecture for periodic Hamiltonian system.

Floer extablished $HF(L_1, L_2)$ satisfying (i)(ii)(iii)(iv) above, under the assumption, $\pi_2(X, L) = 0$ and that there exists Hamiltonian diffeomprphisms φ_i such that $L_i = \varphi_i(L)$. This assumption is rather restrictive.

Y.-G. Oh [59] relaxed this condition to 'L is monotone[3] and the minimal Maslov number[4] is not smaller than 3.'

Actually the Floer homology that satisfies all of (i)(ii)(iii)(iv) above can not exist. In fact for any compact Lagrangian submanifold L of $\mathbb{C}^n$ we can find a Hamiltonian diffeomorphism φ such that $L \cap \varphi(L) = \emptyset$. Hence (2.1) can not hold. ($\mathbb{C}^n$ is non compact. However we can take $X = \mathbb{C}P^n$ instead. In fact the above mentioned

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[1]The notion of Hamitonian diffeomorphim is defined as follows. Let $H : [0, 1] \times X \to \mathbb{R}$ be a smooth funciton. We put $H_t(x) = H(t, x)$. A time dependent Hamiltonian vector field associated to it is a vector V_{H_t} that satisfies $\omega(V, V_{H_t}) = dH_t(V)$, for any vector field V. By the non-degeneracy of symplectic form ω the vector field V_{H_t} is determined uniquely by this condition. We define $\varphi_t^H : X \to X$ by $\varphi_0^H(x) = x$, $(d/dt)(\varphi_0^H(x))|_{t=t_0} = V_{H_{t_0}}(\varphi_{t_0}^H(x))$. A Hamiltonian diffeomorphism φ is a diffeomorpism such that $\varphi = \varphi_1^H$ for some H. It is well known that Hamiltonian diffeomorphism preserves a symplectic form.

[2]Note however that this claim is not correct in general. Arnold certainly did NOT conjecture it of course.

[3] The monotonicity is the condition that the two homomorphisms $H_2(M, L; \mathbb{Z}) \to \mathbb{R} : \beta \mapsto \int_\beta \omega$ (where ω is the symplectic form), and the Maslov index μ (that is a kind of relative version of Chern number) are proportional to each other.

[4]The minimal Maslov number is the Maslov index of the nonzero holomorphic disc with smallest Maslov index.

Hamiltonian diffeomorphism has a compact support and so is extended to $\mathbb{C}P^n$. Thus we still have a counter example to (2.1).)

Our main result on Lagrangian Floer homology which modify (i)(ii)(iii)(iv) is as follows[5]. Items (1)(2)(3) correspond to (ii)(iii)(iv), respectively.

Theorem 2.1. ([27, 28]) *For each spin Lagrangian submanifold L we can associate a set $\mathcal{M}(L)$. For each pair of spin Lagrangian submanifolds L_1, L_2 and $b_i \in \mathcal{M}(L_i)$ we can associate Floer cohomology $HF((L_1, b_1), (L_2, b_2); \Lambda)$. They have the following properties.*

(1) *A symplectic diffeomorphism $\varphi : X \to X$ induces a bijection $\varphi_* : \mathcal{M}(L) \to \mathcal{M}(\varphi(L))$. A pair of Hamiltonian diffeomorphisms $\varphi_i : X \to X$ $(i = 1, 2)$ induces the following isomorphism.*

$$(2.2) \quad \begin{aligned} (\varphi_1, \varphi_2)_* : &HF((L_1, b_1), (L_2, b_2); \Lambda) \\ &\to HF((\varphi_1(L_1), \varphi_{1*}(b_1)), (\varphi_2(L_2), \varphi_{2*}(b_2)); \Lambda). \end{aligned}$$

(2) *If $L_1 = L_2 = L$ we have a spectral sequence E such that $E_2 \cong H(L; \Lambda)$ and*

$$(2.3) \quad E_\infty^p \cong F^p(HF((L, b), (L, b); \Lambda))/F^{p-1}(HF((L, b), (L, b); \Lambda))$$

for an appropriate filtration $F^*(HF((L, b), (L, b); \Lambda))$.

(3) *If L_1 intersects L_2 transversaly, then the rank of $HF(L_1, L_2)$ over Λ is not greater than $\#(L_1 \cap L_2)$.*

The coefficient ring Λ is the *universal Novikov field* ([58]) and is the totality of the following infinite series:

$$(2.4) \quad \sum_{i=0}^{\infty} a_i T^{\lambda_i}.$$

Here a_i are rational numbers and λ_i are real numbers. We assume λ_i is strictly increasing with respect to i and $\lim_{i \to \infty} \lambda_i = \infty$.

If we assume moreover λ_i are rational numbers then the totality of such a series is puiseux series ring[6] and so is a field. We can define a T- adic non Archimedean norm on Λ and Λ is complete with respect to this norm[7].

The set $\mathcal{M}(L)$ may be empty. In that case Theorem 2.1 does not contain any interesting informaiton. In other words the statement of Theorem 2.1 itself would be rather obvious. (We may simply define $\mathcal{M}(L)$ is the empty set always. Then all the claims clearly hold.) So to obtain some nontrivial consequence from Theorem 2.1 we need to find a condition for $\mathcal{M}(L)$ to be nonempty, or the spectral sequence (2) to degenerate. We next describe such results. Those results are easier to state after we slightly generalize Theorem 2.1.

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[5]In Theorem 2.1 existence of spin structure is assumed. We may relax it to the existence of relative spin structure.

[6]When we replace the condition $a_i \in \mathbb{Q}$ by $a_i \in \mathbb{C}$ then its becomes the algebraic closure of the formal power series ring over $\mathbb{C}$.

[7]In [28], we introduced one more formal variable e and use a graded ring Λ_{nov}. In this article we do not use e. In that case Floer homology is only $\mathbb{Z}_2$ graded.

The set $\mathcal{M}(L)$ is a subset $\mathcal{M}_{\mathrm{weak,def}}(L)$[8], and is described by the map

$$\pi : \mathcal{M}_{\mathrm{weak,def}}(L) \to H(X; \Lambda_+), \quad \mathfrak{PO} : \mathcal{M}_{\mathrm{weak,def}}(L) \to \Lambda_+$$

as, $\mathcal{M}(L) = \pi^{-1}(0) \cap \mathfrak{PO}^{-1}(0)$. The function $\mathfrak{PO}$ is called the *potential function*. Here Λ_0 is the set of formal sums (2.4) such that $\lambda_i \geq 0$. It becomes a ring. Λ_+ is its maximal ideal and consists of elements such that $\lambda_i > 0$. The Floer cohomology $HF((L_1, b_1), (L_2, b_2); \Lambda)$ is defined under the condition for $b_i \in \mathcal{M}_{\mathrm{weak,def}}(L_i)$ that $\pi(b_1) = \pi(b_2)$, $\mathfrak{PO}(b_1) = \mathfrak{PO}(b_2)$. It satisfies (i)(ii)(iii)[9].

Theorem 2.2. *If the inclusion induced map $H^*(X; \mathbb{Q}) \to H^*(L; \mathbb{Q})$ is surjective for all even $*$, then $\mathcal{M}_{\mathrm{weak,def}}(L)$ is nonempty.*

We consider the case $L_1 = L_2 = L$, $b_1 = b_2$ in the next theorem.

Theorem 2.3.   (1) *The image of the differential of the spectral sequence in*
           (1) *Theorem 2.1 is contained in the Poincaré dual to the kernel of the map*
           $H_*(L) \to H_*(X)$ induced by the inclusion. Especially under the assumption
           of Theorem 2.2, the spectral sequence of Theorem 2.1 degenerates
     (2) *If the Maslov index of all the discs $(D^2, \partial D^2) \to (M, L)$ is nonnegative then*
           the fundamental class of L becomes an nonzero element of E_∞ In particular
           the Floer cohomology $HF((L, b), (L, b); \Lambda)$ is nonzero

The assumptions of the above theorems are rather restrictive. The condition for the set $\mathcal{M}(L)$ to be nonempty or the Floer cohomology to be nonzero, is closely related to the symplectic topology of L. So it is hard to describe it in term of the topology of L only. Theorem 2.1 can be regarded as a background result to study the symplectic topology of L using Floer cohomology.

3. EXAMPLE : TORIC MANIFOLD

The calculation of Floer cohomology is a difficult problem. Especially the Lagrangian Floer cohomology is hard to calculate and we had only sporadic calculations of it, for a long time. Recently a systematic calculation becomes possible in the case of toric manifolds. Let us explain an example before explaining the proof of Theorem 2.1.

A *toric manifold X* is a real $2n$ dimensional manifold on which real n-dimensional torus T^n acts, so that it admits a map $\pi : X \to \mathbb{R}^n$ that is called the *moment map*. The fibers of π are the T^n orbits, and the image of π is a convex polygon in $\mathbb{R}^n$. We put $\pi(X) = P$. For each $u \in \mathrm{Int}\, P$ the fiber $\pi^{-1}(p) = L(u)$ is diffeomorphic to an n-dimesional torus. Its Floer cohomology $HF((L(u), b), (L(u), b); \Lambda)$ is calculated. In this section we explain a part of this result. In the case of $L = L(u)$, the set $\mathcal{M}_{\mathrm{weak}}(L(u))$ (that is an element b of $\mathcal{M}_{\mathrm{weak,def}}(L(u))$ so that $\pi(b) = 0$) contains $H^1(L(u); \Lambda_0)$ (See [30]). We restrict b to this subset. We then have a map $\mathfrak{PO} : H^1(L(u); \Lambda_0) \to \Lambda_0$[10].

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[8]The set $\mathcal{M}(L)$ is the set of the gauge equivalence classes of the solutions of (4.2). We relax the equation (4.2) to $\sum \mathfrak{m}_k(b^k) \equiv 0 \mod [L]$ and $\mathcal{M}_{\mathrm{weak}}(L)$ is the set of gauge equivalence classes of its solutions. The A_∞ structure of $H(L; \Lambda_0)$ is deformed by an element of $H^{even}(X; \Lambda_+)$ (the map (8.1)), the equation (4.2) (or $\sum \mathfrak{m}_k(b^k) \equiv 0 \mod [L]$)is deformed accordingly. The union of the gauge equivalence classes of their solutions is $\mathcal{M}_{\mathrm{weak,def}}(L)$.

[9](2) holds in case $L_1 = L_2$, $b_1 = b_2$.

[10]In the last section we used Λ_+ in place of Λ_0. This is related to the fact that in case of $\mathbb{R}$ coefficient and M is a real toric manifold, we can slightly generalized the definition of $\mathcal{M}_{\mathrm{weak}}(L)$

In the case of toric manifold, the potential function $\mathfrak{PO}$ is defined as follows. We fix a basis $(\mathbf{e}_i)$ of $H^1(L(u); \mathbb{Z})$, and put $\partial\beta = \sum_{i=1}^n (\partial_i\beta)\mathbf{e}_i$ for $\beta \in H_2(X, L(u); \mathbb{Z})$. (Here $\partial_i\beta$ is an integer.) We describe an element b of $H^1(L(u); \Lambda_0)$ by using the dual basis $(\mathbf{e}_i^*)$ to $(\mathbf{e}_i)$ as $b = \sum_{i=1}^n x_i\mathbf{e}_i^*$. Then,

$$(3.1) \qquad \mathfrak{PO}(b) = \mathfrak{PO}(x_1, \ldots, x_n) = \sum_\beta T^{\beta\cap\omega} \exp\left(\sum_{i=1}^n x_i\partial_i\beta\right) n(\beta).$$

Here, $n(\beta) \in \mathbb{Q}$ is defined roughly as follows. We fix a point $p \in L(u)$. Then $n(\beta)$ is the number of pseudo-holomorphic map $(D^2, \partial D^2) \to (X, L)$ that contains p, and is of homology class β.

Theorem 3.1. *If the gradient vector of $\mathfrak{PO}$ is zero at $b \in H^1(L(u); \Lambda_0)$, then the Floer cohomology is isomorphic to the singular cohomology. Namely:*

$$HF((L(u); b), (L(u); b)); \Lambda) \cong H(T^n; \Lambda).$$

Otherwise,

$$HF((L(u); b), (L(u); b)); \Lambda) = 0.$$

The potential function $\mathfrak{PO}$ is closely related to the Landau-Ginzburg super potential (See [39]). In the case of toric manifold, it is mostly calculated in [30] based on the result of [11]. For example in case $X = \mathbb{C}P^n$ we have $P = \pi(X) = \{(u_1, \cdots, u_n) \mid u_i \geq 0, \sum u_i \leq 1\}$ and the potential function of the fiber $L(u)$ at $u = (u_1, \cdots, u_n)$ is given by:

$$(3.2) \qquad \mathfrak{PO}(x) = \sum_{i=1}^n T^{u_i} e^{x_i} + T^{1-\sum u_i} e^{-\sum x_i}$$

where $x = (x_1, \cdots, x_n) \in \Lambda_0^n \cong H^1(L(u); \Lambda_0)$.

Example 3.2. We consider the case $S^2 = \mathbb{C}P^1 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. (We normalize its symplectic form so that its area is 1.) We consider a T^1-action that consists of rotations around z-axis. Its orbit is parametrized by z-coordinate z_0. We may choose the moment map so that the coordinate u of $P = [0, 1]$ is the area of $\{(x, y, z) \in S^2 \mid z \leq z_0\}$. The complement $\mathbb{C}P^1 \setminus L(u)$ is divided into two discs, one $D_1^2(u)$ contains the south pole, the other $D_2^2(u)$ contains the north pole. We denote the homology class of those discs by β_1, β_2 respectively. These two β's are only the discs which contribute to the right hand side of (3.1). Note $\partial_1\beta_1 = 1$, $\partial_1\beta_2 = -1$, and $\beta_1 \cap \omega = u$, $\beta_2 \cap \omega = 1 - u$. Therefore we have

$$\mathfrak{PO}(x) = T^u e^x + T^{1-u} e^{-x}.$$

In the case of $\mathbb{C}P^n$, there are $n + 1$ homology classes which contribute to the potential function.

The zero of the gradient vector field of the potential function $\mathfrak{PO}$ in (3.2) exists only in case of $u = (1/(n+1), \cdots, 1/(n+1))$. In that case we have $x = (\chi, \cdots, \chi)$, $\chi = 2\pi\sqrt{-1}(1/(n+1) + \text{integer})$.

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in [30] using the idea of [10]. This generalization is actually possible for any X ([26]). Since we discuss only in case of Λ_+ in [28], we only stated the result over Λ_+ coefficient.

The fiber $L(u)$ of this $u = (u_1, \cdots, u_n)$ is called the Clifford torus. It is well known that for all the other fibers $L(u)$ than Clifford torus, there exists a Hamiltonian diffeomorphism φ such that $\varphi(L(u)) \cap L(u) = \emptyset$. Therefore the Floer cohomology $HF((L(u), b), (L(u), b); \Lambda)$ must vanish by Theorem 2.1 (3). Theorem 3.1 is consistent to this fact.

There are several other calculation of the Lagrangian Floer cohomology than toric fibers. For example in [29], we studied how the moduli space of pseudo-holomorphic discs changes by the Lagrangian surgery and perform some calculations using it.

4. A_∞ STRUCTURE

Floer cohomology has a ring structure. Moreover it is an A_∞ algebra. The set $\mathcal{M}(L)$ in Theorem 2.1 can be regarded as a formal scheme defined by those structures. We need to use A_∞ algebra for the proof of Theorem 2.1 also. In this section we discuss A_∞ structure in Floer theory. We restrict ourselves to the case of A_∞ algebra associated to a single Lagrangian submanifold.

A $\mathbb{Z}_2$ graded complete Λ_0 module C is said to be a *filtered A_∞ algebra*, if there exists a series of degree one continuous Λ_0 homomorphisms

$$\mathfrak{m}_k : C[1] \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} C[1] \to C[1],$$

for $k = 0, 1, \cdots$. (Here the left hand side is the tensor product of k copies of $C[1]$.) such that the relation

$$(4.1) \qquad \sum_{k=1}^{n-1} \sum_{i=1}^{k-i+1} (-1)^* \mathfrak{m}_{n-k+1}(x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_i, \cdots, x_{i+k-1}) \otimes \cdots \otimes x_n) = 0,$$

holds. ($* = \deg x_1 + \cdots + \deg x_{i-1} + i - 1$.) (More precisely we assume some complementary conditions such as $\mathfrak{m}_0(1) \equiv 0 \mod \Lambda_+$.) Here $C[1]$ is a parity change of C, that is $C[1]^1 = C^0$ and $C[1]^0 = C^1$. (Note C is $\mathbb{Z}_2$ graded.) (4.1) is called the A_∞ *relation*.

The operator $\mathfrak{m}_k$ defines a coderivation by:

$$\widehat{\mathfrak{m}}_k : x_1 \otimes \cdots \otimes x_n \mapsto \pm x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_i, \cdots, x_{i+k-1}) \otimes \cdots \otimes x_n.$$

We put: $\hat{d} = \sum_{k=0}^\infty \widehat{\mathfrak{m}}_k$. Then (4.1) is equivalent to the formula $\hat{d} \circ \hat{d} = 0$.

Theorem 4.1. *For each (relatively) spin Lagrangian submanifold L, its singular cohomology $H(L; \Lambda_0)$ over Λ_0 has a structre of filtered A_∞ algebra. This filtered A_∞ algebra is invariant of the symplectic diffeomorphism type of (M, ω, L) up to isomorphism.*

We next describe a relation between Theorem 4.1 and Theorem 2.1. Let $b \in H^{odd}(L; \Lambda_+)$. We consider the equation[11].

$$(4.2) \qquad\qquad \mathfrak{m}_0(1) + \sum_{k=1}^\infty \mathfrak{m}_k(b^{\otimes k}) = 0.$$

Let us consider the case $\mathfrak{m}_k = 0$ for $k \neq 1, 2$. We put $\mathfrak{m}_1 = \pm d$, $\mathfrak{m}_2(x, y) = \pm x \wedge y$. Then (4.2) becomes the Maurer-Cartan equation $db + b \wedge b = 0$. Maurer-Cartan equation gives a condition for the connection $\nabla = d + b$ to be flat. (Namely it is

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[11]The right hand side of this equation is infinite sum. However since we assumed $b \equiv 0 \mod \Lambda_+$ it converges in T-adic topology.

equivalent to $\nabla \circ \nabla = 0$.) Therefore Maurer-Cartan equation describes the moduli space of flat bundles[12].

The set of gauge equivalence classes[13] of (4.2) is the set $\mathcal{M}(L)$. For $[b_1], [b_2] \in \mathcal{M}(L)$, $x \in H(L; \Lambda_0)$ we define

$$\delta_{b_1, b_2}(x) = \sum_{k_1, k_2} \mathfrak{m}_{k_1 + k_2 + 1}(b_1^{\otimes k_1} \otimes x \otimes b_2^{\otimes k_2}).$$

Then (4.1), (4.2) implies $\delta_{b_1, b_2} \circ \delta_{b_1, b_2} = 0$. The cohomology group of this boundary operator δ_{b_1, b_2} is Floer cohomology $HF((L, b_1), (L, b_2); \Lambda_0)$[14].

To prove Theorem 4.1 we define operators $\mathfrak{m}_k$ on the cohomology group. The definition is based on the moduli space of pseudo-holomorphic discs and is performed as follows.

We first take an almost complex structure $J : TX \to TX$ on X that is an $\mathbb{R}$ linear map such that $J \circ J = -1$. We assume that J is compatible with ω, namely $g(V, W) = \omega(V, JW)$ is a Riemannian metric. We say a map $u : \Sigma \to X$ is *pseudo-holomorphic* or J-holomorphic if the formula:

$$J(du(V)) = du(j(V))$$

is satisfied. Here $j : T\Sigma \to T\Sigma$ is a complex structure of Σ. Let $\beta \in H_2(X, L; \mathbb{Z})$. For $k \geq 0$ we define the set $\widetilde{\mathcal{M}}_{k+1}^{\mathrm{reg}}(L; \beta)$ to be the totality of all pairs[15] $(u, \vec{z})$ of maps $u : D^2 \to X$ and $\vec{z} = (z_0, \cdots, z_k)$ that have the following 6 properties.

(1) u is pseudo-holomorphic.
(2) $u(\partial D^2) \subset L$.
(3) The relative homology class of u is β.
(4) $z_i \in \partial D^2$.
(5) If $i \neq j$ then $z_i \neq z_j$.
(6) $z_0, z_1, \cdots, z_k$ respects the counterclockwise cyclic order of ∂D^2.

The group of all biholomorphic isomorphisms of D^2 is isomorphic to $PSL(2; \mathbb{R})$. It acts on $\widetilde{\mathcal{M}}_{k+1}^{\mathrm{reg}}(L; \beta)$ by $g \cdot (u, \vec{z}) = (g \cdot u, g\vec{z})$, $(g \cdot u)(z) = u(g^{-1}(z))$, $g\vec{z} = (gz_0, \cdots, gz_k)$.

We denote by $\mathcal{M}_{k+1}^{\mathrm{reg}}(L; \beta)$ the quotient space of this action. We define a quotient topology on it. It has a compactification which is Hausdorff. To obtain a compactification, we use a generalization of the notion of the stable map ([47]) to the case of bordered Riemann surface. We denote the compactification by $\mathcal{M}_{k+1}(L; \beta)$. The map $ev = (ev_0, \cdots, ev_k) : \widetilde{\mathcal{M}}_{k+1}^{\mathrm{reg}}(L; \beta) \to L^{k+1}$ is defined by $ev_i(u, \vec{z}) = u(z_i)$. This map is invariant of $PSL(2; \mathbb{R})$ action and extends to the compactification. In other words we have a map $ev : \mathcal{M}_{k+1}(L; \beta) \to L^{k+1}$. We call it the *evaluation map*. Let us assume the following for simplicity.

(*) $\mathcal{M}_{k+1}(L; \beta)$ is a compact oriented manifold.

We remark that this assumption is rarely satisfied. We discuss this points in Sections 5-6. In case (*) is satisfied, the dimension of the moduli space $\mathcal{M}_{k+1}(L; \beta)$

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[12]In (4.2) the element b is a cohomology classes. In Maurer-Cartan equation b is a differential form rather than de Rham cohomology class. See three lines right before Remark 6.4 about this point.

[13]We omit the definition of gauge equivalence. See [28] section 4.3.

[14]We use Λ instead of Λ_0 in Theorem 2.1. This is because (2) holds only over this coefficient ring.

[15]$D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$.

is given by:

$$(4.3) \qquad \dim \mathcal{M}_{k+1}(L; \beta) = n + (k+1) + \mu(\beta) - 3 = n + k - 2 + \mu(\beta).$$

Here $\mu : H_2(X, L; \mathbb{Z}) \to \mathbb{Z}$ is a homomorphism called Maslov index.

In case (*) is satisfied, we define $\mathfrak{m}_{k,\beta} : (H^*(L; \mathbb{Z}))^{k\otimes} \to H^*(L; \mathbb{Z})$ by

$$(4.4) \qquad \begin{aligned} &\mathfrak{m}_{k,\beta}(P_1, \cdots, P_k) \\ &= PD_L(ev_{0*}(PD_{\mathcal{M}_{k+1}(L;\beta)}((ev_1, \cdots, ev_k)^*(P_1 \times \cdots \times P_k)))). \end{aligned}$$

Here PD is the Poincaré duality map. The first one is $PD_L : H_d(L) \to H^{n-d}(L)$, and the second one is

$$PD_{\mathcal{M}_{k+1}(L;\beta)} : H^d(\mathcal{M}_{k+1}(L;\beta)) \to H_{n+k-2+\mu(\beta)-d}(\mathcal{M}_{k+1}(L;\beta)).$$

Then we define $\mathfrak{m}_k$ by

$$(4.5) \qquad \mathfrak{m}_k = \sum_{\beta} T^{\beta \cap \omega} \mathfrak{m}_{k,\beta}.$$

Here $\beta \cap \omega = \int_u \omega$. ($u$ is a relative cycle representing $\beta \in H_2(X, L; \mathbb{Z})$.) Since ω is 0 on L, the integral $\int_u \omega$ depends only on the relative homology class β and is independent of u. We can use Gromov compactness (that is the compactness of the moduli space of pseudo-holomorphic maps u so that $\int_u \omega$ is smaller than a fixed number) to show the convergence of $\mathfrak{m}_k$ with respect to the non-Archimedean topology of Λ_0.

5. Kuranishi structure

In the last section, we assumed (*) to define $\mathfrak{m}_{k,\beta}$. The assumption (*) is rarely satsified. For example it is never satisfied in case $\beta = 0$, $k > 2$. As a consequence we *cannot* use (4.4) itself to define $\mathfrak{m}_{k,\beta}$. We use the method of virtual fundamental chain[16] and chain level intersection theory to overcome this trouble. We discuss the former in this section and the later in the next section. We need to study the following two points.

(A) In general, an element of $\mathcal{M}_{k+1}(L; \beta)$ has a singularity. Moreover it may have nontrivial automorphism.
(B) The moduli space $\mathcal{M}_{k+1}(L; \beta)$ has a (codimension 1) boundary. So even in the situation (A) does not occur, it does not determine a cycle.

Let us first discuss the point (A). In order to isolate this point from (B), we study the following situation, which is slightly different from one in the last section. We consider a (real) 6 dimensional symplectic manifold X with $c_1(X) = 0$. (We use a compatible almost complex structure J to define the first Chern class $c_1(X)$.) For each $\alpha \in H_2(X; \mathbb{Z})$, we consider the set of the pseudo-holomorphic maps $u : S^2 \to X$ of homology class α. We identify the maps which are transformed by the group of automorphisms of $S^2 (= PSL(2; \mathbb{C}))$ and compactify the set of equivalence classes of this identification. Then we obtain the moduli space $\mathcal{M}(\alpha)$. In case $\mathcal{M}(\alpha)$ is a manifold, we calculate its dimension using $c_1(X) = 0$ and obtain $\dim \mathcal{M}(\alpha) = 0$. In other words, $\mathcal{M}(\alpha)$ is an oriented 0 dimensional compact manifold. So its order

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[16]The notion of virtual fundamental class is a natural generalization of the notion of the fundamental class $[M] \in H_n(M; \mathbb{Z})$ of an oriented manifold M. In our situation, we do not take the homology class but use cycle or chain. So we need the notion of virtual fundamental *chain*.

(counted with sign) makes sense. This is the simplest case of Gromov-Witten invariant $GW(\alpha)$.

The problem (A) in this case is the problem to justify 'the order counted with sign' when $\mathcal{M}(\alpha)$ may be not a manifold.

The theory of virtual fundamental class resolves this problem. In the case of $GW(\alpha)$ there are two methods to work it out. One is based on algebraic geometry ([53, 7]) and the other is based on differential geometry ([33, 65, 54, 73]). In the case when Lagrangian submanifold is included, we can not use the method of algebraic geometry. So we explain the method based on differential geometry.

We use the notion of Kuranishi structure to define virtual fundamental class. A Kuranishi structure on a topological space $\mathcal{M}$ is, roughly speaking, a system to give the way to represent the space $\mathcal{M}$ locally as the zero set of an equation (on a finite dimensional space) so that they are glued in a consistent way. Namely for each $u \in \mathcal{M}$, we represent a neighborhood of u in $\mathcal{M}$ as the set of the solutions of m equations $s_{u,i}(x_1, \cdots, x_n) = 0$ $(i = 1, \cdots, m)$ of n-variables. The main idea is to include not only the solution set but also the equation itself as a part of the structure. When we change the base point u the equation may change. However only the following 2 kinds of changes are allowed.

(1) We increase the number of equations and variables by the same number, say ℓ. The equation itself is modified by the following trivial way.

$$s_{u,i}(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+\ell}) = s_{u,i}(x_1, \cdots, x_n) \qquad i = 1, \cdots, m$$
$$s_{u,i+m}(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+\ell}) = x_{i+n} \qquad i = 1, \cdots, \ell$$

(2) The coordinate transformation of the variables x_i and the linear transformation of the equations. We may allow the linear transformations of the equation to depend on x_i. Namely we allow the following transformation $(s_u; x_1, \cdots, x_n) \mapsto (s_v; y_1 \cdots, y_n)$.

$$s_{v,j}(y_1, \cdots, y_n) = \sum_{i=1}^{m} g_{ji}(x_1, \cdots, x_n) s_{u,i}(x_1, \cdots, x_n), \quad y_i = y_i(x_1, \cdots, x_n).$$

(3) We include the process to divide the whole structure by a finite group.

We include the map s_u in the data defining Kuranishi structure. So the multiplicity is determined from the Kuranishi structure. For example, in case $n = m = 2$, $s(x, y) = (x^2 - y^2, 2xy)$, the solution set $s^{-1}(0)$ consists of one point 0. This is the same as the case $s(x, y) = (x, y)$. However they are different as Kuranishi structures. The virtual fundamental class (which we define later) is a 0 dimensional homology class, that is a number, in this case. In case $s(x, y) = (x^2 - y^2, 2xy)$, it is 2. In case $s(x, y) = (x, y)$, it is 1.

Let us explain the notion of Kuranishi structure in more detail and also explain the way how it appears in the study of pseudo-holomorphic curve.

When we study moduli spaces using differential geometry, we study the set of the solutions of a differential equation, which is elliptic modulo the action of the 'gauge group'. Here we study the nonlinear Cauchy-Riemann equation:

(5.1) $$J \circ du = du \circ j.$$

It linearization is a differential operator:

$$D_u \overline{\partial} : \Gamma(S^2; u^* TX) \to \Gamma(S^2; u^* TX \otimes \Lambda^{01}(S^2)),$$

where $\Lambda^{01}(S^2)$ is the complex line bundle of $(0,1)$ forms on S^2, and Γ denotes the set of smooth sections.

If $D_u\overline{\partial}$ is surjective then we can use implicit function theorem to show that the solution set of (5.1) is a smooth manifold in a neighborhood of u. It is finite dimensional by ellipticity. We denote a neighborhood of u in the solution set by $V(u)$.

Our moduli space is the quotient space of the set of solutions of (5.1) by the $PSL(2;\mathbb{C})$-action. If the map u is nontrivial, the set of $g \in PSL(2;\mathbb{C})$ satisfying $g \cdot u = u$ is finite. We denote it by $\Gamma(u)$. Then the neighborhood of $[u]$ in our moduli space $\mathcal{M}(\alpha)$ is $V(u)/\Gamma(u)$. The space $V(u)$ is a manifold if $D_u\overline{\partial}$ is surjective. Therefore $V(u)/\Gamma(u)$ is a quotient of a manifold by a finite group. Satake introduced the notion of a space which is locally a quotient of a manifold by a finite group. It is nowadays called an *orbifold*. Compact and oriented orbifold without boundary carries a fundamental class in the same way as manifold. Thus Problem (A) does not occur if $\mathcal{M}(\alpha)$ is an orbifold.

The problem (A) occurs when $D_u\overline{\partial}$ is not surjective. The local theory of such a moduli space is studied by Kuranishi in the case of the moduli space of complex manifolds. We apply it in the case of the moduli space of pseudo-holomorphic curves.

We take a finite dimensional linear subset $E(u) \subset \Gamma(S^2; u^*TX \otimes \Lambda^{01}(S^2))$ such that the sum of the image of $D_u\overline{\partial}$ and $E(u)$ generates $\Gamma(S^2; u^*TX \otimes \Lambda^{01}(S^2))$ as a vector space. This is possible since $D_u\overline{\partial}$ is a Fredholm operator.

We may choose a family of isomorphisms $\Gamma(S^2; u^*TX \otimes \Lambda^{01}(S^2)) \cong \Gamma(S^2; v^*TX \otimes \Lambda^{01}(S^2))$ depending smoothly on v. Then we may regard $E(u)$ as a subset of $\Gamma(S^2; v^*TX \otimes \Lambda^{01}(S^2))$. Now we replace the equation (5.1) by

(5.2) $J \circ dv - dv \circ j \in E(u).$

We can then apply implicit function theorem to it. We regard the element of $\Gamma(S^2; v^*TX \otimes \Lambda^{01}(S^2))$ as a map which associate to $x \in S^2$ an anti-complex-linear map $T_xS^2 \to T_{v(x)}X$. We can make sense (5.2) by this identification.

Therefore the set of solutions of (5.2) is a finite dimensional manifold, which we denote by $V(u)$. We define a map $s_u : V(u) \to E(u)$ by $v \mapsto J \circ dv - dv \circ j$. Finally we put

$$\Gamma(u) = \{g \in PSL(2;\mathbb{C}) \mid u \circ g = u\}.$$

(This is the automorphism group.) The group $\Gamma(u)$ is a finite group.

We take $E(u)$ so that it is $\Gamma(u)$ invariant. Then $\Gamma(u)$ acts on $V(u)$ and $E(u)$, so that s_u is $\Gamma(u)$-invariant. We thus obtain the following:

A neighborhood $V(u)$ in a complex vector space of finite dimension. A linear and effective action of a finite group $\Gamma(u)$ on it. A finite dimensional representation of $\Gamma(u)$ on $E(u)$. Moreover we have a $\Gamma(u)$-equivariant map $s_u : V(u) \to E(u)$ and a homeomorphism $\psi_u : s_u^{-1}(0)/\Gamma(u) \to \mathcal{M}(\alpha)$ onto a neighborhood of u.

We call $(V(u), E(u), \Gamma(u), s_u, \psi_u)$ a *Kuranishi chart*.

We can define the notion of coordinate change between Kuranishi charts in the same way as the definition of manifold as follows.

A *coordinate change* from a Kuranishi chart $(V(u_1), E(u_1), \Gamma(u_1), s_{u_1}, \psi_{u_1})$ to another Kuranishi chart $(V(u_2), E(u_2), \Gamma(u_2), s_{u_2}, \psi_{u_2})$ consists of a $\Gamma(u_1)$-invariant open subset $V(u_1, u_2) \subset V(u_1)$, a group homomorphism $\phi_{u_2u_1} : \Gamma(u_1) \to \Gamma(u_2)$, $\phi_{u_2u_1}$-equivariant smooth embedding $\varphi_{u_2u_1} : V(u_1, u_2) \to V(u_2)$, a $\phi_{u_2u_1}$-equivariant

smooth embedding of vector bundles $\hat{\varphi}_{u_2 u_1} : V(u_1, u_2) \times E(u_1) \to V(u_2) \times E(u_2)$ over $\varphi_{u_2 u_1}$ such that the following compatibility condition is satisfied:

$$s_{u_2} \circ \varphi_{u_2 u_1} = \hat{\varphi}_{u_2 u_1} \circ s_{u_1}, \qquad \psi_{u_2} \circ \varphi_{u_2 u_1} = \psi_{u_1} \quad \text{holds on } s_{u_1}^{-1}(0) \cap V(u_1, u_2) .$$

We can define the compatibility conditions between coordinate changes in the same way as the definition of manifolds.

A paracompact Hausdorff space is said to have a *Kuranishi structure* when it is covered by Kuranishi charts so that coordinate changes are defined among them and the compatibility between them is satisfied. We assume one more condition (5.3) explained later. We call $E(u)$ the *obstruction bundle*, and s_u the *Kuranishi map*.

The biggest difference between manifold structure and Kuranishi structure lies in the fact that we do not assume the embedding $\varphi_{u_2 u_1} : V(u_1, u_2) \to V(u_2)$ to be a local homeomorphism. Especialy the dimension of $V(u_1)$ may not be equal to the dimension of $V(u_2)$. We assume the following condition instead.

(5.3) $\qquad\qquad \dim V(u) - \operatorname{rank} E(u)$ is independent of u.

We call this difference the *dimension* (or *virtual dimension*) of our Kuranishi structure.

We need one more condition to define the virtual fundamental class of the Kuranishi structure. We consider the normal bundle $N_{V(u_1, u_2)} V(u_2)$ of our smooth embedding $\varphi_{u_2 u_1} : V(u_1, u_2) \to V(u_2)$. The compatibility condition implies that s_{u_2} induces the following linear map between vector bundles.

$$ds_{u_2} : N_{V(u_1, u_2)} V(u_2) = \frac{\varphi_{u_2 u_1}^* TV(u_2)}{TV(u_1)} \to \frac{\varphi_{u_2 u_1}^* E(u_2)}{E(u_1)}.$$

We say that our Kuranishi structure has a *tangent bundle* if ds_{u_2} is an isomorphism. From now on we consider only the Kuranishi structure with tangent bundle. We can define the notion of orientability or orientation of Kuranishi structure with tangent bundle.

As we mentioned before, a Kuranishi structure on $\mathcal{M}$ may be regarded as a system which assigns a way to represent $\mathcal{M}$ as the solution set of the equation $s_u = 0$ on a neighborhood of each point u, so that it is consistent when we move u. The notion of a coordinate of a space with singularity is classical in algebraic geometry. The definition of algebraic variety or scheme is bases on such a notion[17]. In complex analytic category, there is a similar notion such as analytic space or Douady space. Here we work on C^∞ category. As a consequence the following point is different from those.

Scheme is define as a locally ringed space. There is a difficulty to do so here.

For each Kuranishi chart $(V(u), E(u), \Gamma(u), s_u, \psi_u)$ we put $s_u = (s_u^1, \cdots, s_u^m)$. Then we may consider the quotient of the ring of function germs $C_0^\infty(V(u))$ by the ideal $(s_u^1, \cdots, s_u^m)$ generated by $s_u^1, \cdots, s_u^m$ and define a local ring $C_0^\infty(V(u))/(s_u^1, \cdots, s_u^m)$. By moving u we obtain a locally ringed space. However this locally ringed space is a bit hard to study. The biggest problem is that $C_0^\infty(V(u))$ is harder to handle compared to the ring of germs of holomorphic functions or polynomial rings. For example the Krull dimension of $C_0^\infty(V(u))$ is infinite. In particular the Krull dimension of $C_0^\infty(V(u))/(s_u^1, \cdots, s_u^m)$ is not equal to $\dim V(u) - \operatorname{rank} E(u)$.

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[17]Since we include the finite group action our notion of Kuranishi structure corresponds to Deligne-Mumford stack in algebraic geometry.

By this reason we avoid using the ring $C^\infty(V(u))/(s_u^1, \cdots, s_u^m)$ or its localization. Instead we cover our space by the charts which have positive size. Moreover we assumed compatibility among the Kuranishi maps. In the situation when we can use the language of locally ringed space, the compatibility of the Kuranishi maps can be replaced by the condition that the structure sheaf is defined globally.

Remark 5.1. Here is another difference between our case and the case of algebraic geometry or analytic space. Suppse $\dim V = 1$, $\text{rank } E = 1$. We consider the following three cases. '$s(x) = x$', '$s'(x) = x^3$', '$s''(x) = e^{-1/x}(x > 0), -e^{1/x}(x < 0)$'. The multiplicity of s and s' at origin is 1 and 3 respectively in the usual sense of algebraic geometry. So we might define the virtual fundamental cycle of $(\mathbb{R}, \mathbb{R}, \{1\}, s)$ to be 1, and of $(\mathbb{R}, \mathbb{R}, \{1\}, s')$ to be 3. However we define the virtual fundamental cycle of $(\mathbb{R}, \mathbb{R}, \{1\}, s')$ to be 1. This is justified by considering $s'_\epsilon(x) = x^3 + \epsilon x$ for sufficiently small ϵ. (Note x is a real variable.)

In the case of $(\mathbb{R}, \mathbb{R}, \{1\}, s'')$, the multiplicity might be infinity if we consider the analogy of algebraic geometry. However the virtual fundamental cycle $(\mathbb{R}, \mathbb{R}, \{1\}, s'')$ is 1.

Let us define the virtual fundamental cycle of the space $\mathcal{M}$ with oriented Kuranishi structure. A map $f : \mathcal{M} \to Y$ to a topological space Y is said to be *strongly continuous* if f is extended to its Kuranishi neighborhood $f_u : V(u) \to Y$ for each $u \in \mathcal{M}$ and they are compatible with $\varphi_{u_2 u_1}$. We omit the detail of the compatibility condition. The virtual fundamental class is an element of $H_{\dim \mathcal{M}}(Y)$.

We first consider the case when $\Gamma(u)$ is always trivial. Then we can deform the Kuranishi map s_u on each of the Kuranishi chart $(V(u), E(u), \Gamma(u), s_u, \psi_u)$ to obtain $\mathfrak{s}_u$, such that $\mathfrak{s}_u^{-1}(0)$ becomes a $\dim \mathcal{M}$ dimensional manifold. Using the compatibility of the Kuranishi charts we can take $\mathfrak{s}_u$ so that they are compatible with the coordinate change in a suitable sense. Then the zero sets, $\mathfrak{s}_u^{-1}(0)$ are glued to define a $\dim \mathcal{M}$ dimensional manifold. The strong continuity implies that f defines a map from this manifold to Y. Therefore $f_*[\cup_u \mathfrak{s}_u^{-1}(0)] \in H_{\dim \mathcal{M}}(Y; \mathbb{Z})$ is defined.

In case $\Gamma(u)$ is nontrivial, it is impossible to find a $\Gamma(u)$-invariant $\mathfrak{s}_u$ that is transversal to 0, in general. In this case, we use multisection instead. We first take $\mathfrak{s}_u$ which is transversal to 0. Then we consider the totality of the $\Gamma(u)$ orbits of it. Namely we consider $\{\gamma \cdot \mathfrak{s}_u \mid \gamma \in \Gamma(u)\}$. This is $\Gamma(u)$ invariant as a set. We regard it as a multivalued section. Then its zero set, that is the set of all points where at least one of the $\#\Gamma(u)$-branches of this multivalued section is zero, carries a fundamental cycle as follows. We first triangulate this zero set. Then for each simplex Δ_a of top dimension, we define a wight $m_a \in \mathbb{Q}$ as follows. We divide the number of the branches which becomes zero on it by the order $\#\Gamma(u)$. The weight is this ratio. $\sum m_a(\Delta_a, f)$ is a singular chain of Y. We can show that it is a cycle of Y. This is the *virtual fundamental class* $\in H(Y; \mathbb{Q})$.

In case $\dim \mathcal{M} = 0$ we do not need to specify Y to define the virtual fundamental class as a degree 0 homology class. It is a rational number. We regard it as the 'number of the points of $\mathcal{M}$'.

The moduli space $\mathcal{M}(\alpha)$ of pseudo-holomorphic curves has an oriented Kuranishi structure. In case $[u] \in \mathcal{M}(\alpha)$ is represented by a map $u : S^2 \to X$ we explained the way to find its Kuranishi neighborhood already. We need a compactification and the case $u : S^2 \to X$ corresponds to the case when u is in the interior of $\mathcal{M}(\alpha)$. To define a Kuranishi structure on $\mathcal{M}(\alpha)$ we need to define a Kuranishi

chart on a neighborhood of the point u at infinity. Such u is a map from a singular Riemann surface Σ with normal crossing singularity. Starting with $u : \Sigma \to X$, we can define a family of pseudo-holomorphic maps from the normalization of Σ. This is an important topic in the theory of pseudo-holomorphic curve and is called the gluing. (A similar procedure had been studied in gauge theory. It was initiated by Taubes to prove an existence theorem of self-dual connection on 4 manifolds.) We use the theory of glueing to construct a Kuranishi chart on a neighborhood of a point u at infinity.

Remark 5.2. The notion of Kuranishi structure was introduced at the year 1996 by Fukaya-Ono. The construction of virtual fundamental class by the differential geometric method was done independently by Li-Tian, Ruan, Siebert in the same year. Fukaya-Ono used the notion of multisection explained above. Ruan used de Rham cohmology and Li-Tian used the notion of normal cone which had been used in algebraic geometry in a related but different purpose. Fukaya-Ono and Ruan used finite dimensional approximation. Li-Tian did not take finite dimensional approximation and studied infinite dimensional space directly.

After the turn of the century, a notion called polyfold is proposed by Hofer-Wysocki-Zehnder. The theory of polyfold follows ours in the two points below.

(1) It defines an appropriate class of spaces including various moduli spaces. It associates a virtual fundamental cycle to the spaces in that class in a way independent of the way how such a structure is obtained.

(2) It uses multivalued abstract perturbation[18].

It is also easy to show the following: For any polyfold there is a space with Kuranishi structure which has the same virtual fundamental class.

Therefore the story of Kuranishi structure is applicable to *any* problem to which the story of polyfold is applicable.

It seems that the most novel part of the theory of polyfold is its analytic part. The theory of Kuranishi structure starts at the borderline where analysis is over and topology starts. Namely in the story of Kuranishi structure the construction of Kuranishi structure is left to the study of each of the problems and the abstract theory starts at the point where finite dimensional approximation (the Kuranishi structure) is constructed. On the other hand, in the story of polyfold, one of the main part of the construction that is the glueing is included in the general theory and it formulates the situation where glueing becomes possible. Since the story of polyfold is not yet worked out in detail, we do not discuss it here.

Remark 5.3. As we mentioned before, the notion of Kuranishi structure is a kind of C^∞ analogue of the notion of scheme and stack. Therefore it seems important to define a category of the space with Kuranishi structure[19]. Especially it is important to find a good notion of morphisms between them. The author did not find a good way to do so yet. A difficulty to do so is as follows. A coordinate change of Kuranishi chart should be an example of such morphisms and should be an

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[18] Here abstract perturbation is the method in which, instead of specifying the explicit way to perturb the equation, we consider the moduli space locally as a zero set of an abstract map and perturb it in an abstract way. This is not so much a new idea and actually was used in [12] more than 30 years ago.

[19] From the point of view of analogy with scheme, fiber product is important. The fiber product of spaces of Kuranishi structure over manifold is defined in [28] §A1 and is applied. We remark however that this may not be the fiber product in the sense of category theory

isomorphism. Namely we need to regard the Kuranishi structure (U, E, s) and $(U \times \mathbb{R}^m, E \times \mathbb{R}^m, (s, id))$ to be isomorphic. We can define a map $(U, E, s) \rightarrow (U \times \mathbb{R}^m, E \times \mathbb{R}^m, (s, id))$ in a natural way. However it is hard to find a map in the opposite direction in the usual sense. So we need to localize the category. Several conditions are required for the localization to be well-defined. Those conditions are not trivial to check.

6. Chain level intersection theory

The story described in the last section is the case of pseudo-holomorphic map from closed Riemann surface without boundary, that was establishd in 1996. The theory of Floer homology studies the case of the pseudo-holomorphic map from compact Riemann surface with boundary (bordered Riemann surface). The novel point appearing here is problem (B) in the last section. We discuss it in this section (based on [28] section 7.2).

The main problem is that the moduli space $\mathcal{M}_{k+1}(L; \beta)$ has Kuranishi structure with corners. The notion of Kuranishi structure with corners is defined in the same way as before by including the case when $U(u)$ is an open subset of $[0, \infty)^k \times \mathbb{R}^{n-k}$.

Note a manifold with boundary or corner does not carry a fundamental class. In other words the correspondence by a manifold with boundary or corner does not induce a map between homology groups. By this reason, we need chain level intersection theory.

Remark 6.1. If a strongly continuous map $f : \mathcal{M} \rightarrow Y$ and a subset $Z \subset Y$ satisfy $f(\partial \mathcal{M}) \subset Z$, then we can define a 'relative virtual fundamental cycle' $f_*([\mathcal{M}]) \in H(Y, Z; \mathbb{Q})$ in the same way. However in Lagrangian Floer theory, we need to consider the case when $ev : \mathcal{M}_{k+1}(L; \beta) \rightarrow L^{k+1}$ is the strongly continuous map f. In this case, there does not seem to be a reasonable proper subset of L^{k+1} containing $ev(\partial(\mathcal{M}_{k+1}(L; \beta)))$. The author does not know any example where 'relative virtual fundamental cycle' in the above sense was applied successfully.

Because of the problem we mentioned above, we need to perform the construction of the operator $\mathfrak{m}_k$ in the chain level, in order to construct our A_∞ algebra. The construction is roughly as follows. For each singular chain $\sigma_i : \Delta^{d_i} \rightarrow L$, we take the fiber product

$$(6.1) \qquad \mathcal{M}_{k+1}(L; \beta) \;\; _{ev_1, \cdots, ev_k} \times_{\sigma_1, \cdots, \sigma_k} \prod_{i=1}^{k} \Delta^{d_i}$$

over L^k and triangulate it. Then we regard each of the simplices of the top dimension of (6.1) as a singular chain by the map ev_0. The sum of them is $\mathfrak{m}_{k,\beta}(\sigma_1, \cdots, \sigma_k)$ by definition. It is rather a heavy job to work out its detail[20]. We explain this construction a bit more below.

We denote the fiber product (6.1) as $\mathcal{M}_{k+1}(L; \beta; \sigma_1, \cdots, \sigma_k)$. This space has a Kuranishi structure with corner. Its boundary is decomposed into the sum of the following two types of spaces:

(a) The fiber product between $\Delta^{d_1} \times \cdots \times \Delta^{d_{i-1}} \times \mathcal{M}_{j-i+1}(L; \beta_1; \sigma_i, \cdots, \sigma_j) \times \Delta^{d_{j+1}} \times \cdots \times \Delta^{d_k}$ and $\mathcal{M}_{k+1+i-j}(L; \beta_2)$

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[20]The longest section §7.2 of [28] is devoted to work it out.

(b) $\mathcal{M}_{k+1}(L; \beta; \sigma_1, \cdots, \partial_c \sigma_i, \cdots, \sigma_k)$ Here $\partial_c \sigma_i$ is c-th face of the sigular chain σ_i.

Then we define multi-sections on $\mathcal{M}_{k+1}(L; \beta; \sigma_1, \cdots, \sigma_k)$ and triangulations of its zero set inductively according to the order of k, $\omega \cap \beta$, $\sum(n - d_i)$, so that they are compatible with the above description of the boundaries. It determines the virtual fundamental chains of (6.1) for each of $\beta, \sigma_1, \cdots, \sigma_k$. We thus defined $\mathfrak{m}_{k,\beta}(\sigma_1, \cdots, \sigma_k)$ as a singular chain. We define $\mathfrak{m}_{1,0}$ to be the boundary operator of the singular chain complex. Then $\mathfrak{m}_k$ is defined by (4.5).

We can prove that $\mathfrak{m}_k$ defines an A_∞ structure as follows. The boundary of (6.1) is

$$\mathfrak{m}_{1,0}(\mathfrak{m}_{k,\beta}(\sigma_1, \cdots, \sigma_k)).$$

This is decomposed into (a) and (b) above. The case (a) gives

$$\mathfrak{m}_{k+i-j,\beta_1}(\sigma_1, \cdots, \sigma_{i-1}, \mathfrak{m}_{j-i+1,\beta_2}(\sigma_i, \cdots, \sigma_j), \sigma_{j+1}, \cdots, \sigma_k)$$

and (b) gives

$$\mathfrak{m}_{k,\beta}(\sigma_1, \cdots, \mathfrak{m}_{1,0}(\sigma_i), \cdots, \sigma_k).$$

Thus we have

$$\mathfrak{m}_{1,0}(\mathfrak{m}_{k,\beta}(\sigma_1, \cdots, \sigma_k))$$

$$= \sum_i \pm \mathfrak{m}_{k,\beta}(\sigma_1, \cdots, \mathfrak{m}_{1,0}(\sigma_i), \cdots, \sigma_k)$$

$$+ \sum_{\beta_1+\beta_2=\beta} \sum_{1 \le i \le j \le k} \pm \mathfrak{m}_{k+i-j,\beta_1}(\sigma_1, \cdots, \sigma_{i-1}, \mathfrak{m}_{j-i+1,\beta_2}(\sigma_i, \cdots, \sigma_j), \sigma_{j+1}, \cdots, \sigma_k)$$

This is equivalent to (4.1).

This construction is nontrivial also when we restrict it to the case of $\beta = 0$. In that case it defines a structure of A_∞ algebra on the singular chain complex.

Remark 6.2. We here explained the construction using singular homology. At the time of writing this article, two other constructions are known.

One is to use de Rham cohomology. In this case, the formula (6.1) corresponds to

$$(\rho_1, \cdots, \rho_k) \mapsto (ev_0)_! \left(ev_1^* \rho_1 \wedge \cdots \wedge ev_k^* \rho_k \right).$$

Here ρ_i is a differential form on L and $ev_i^* \rho_i$ is its pull back. $(ev_0)_!$ is the integration along the fiber by the map $ev_0 : \mathcal{M}(L; \beta) \to L$. Even in the case when $\mathcal{M}(L; \beta)$ is a smooth manifold, the map ev_0 may not be a submersion. So the integration along the fiber defines a distributional form that may not be a smooth form. So we can not define the operator $\mathfrak{m}_k$ on the de Rham complex of L in this way. We use smoothing of the distributional form $\mathcal{M}(L; \beta)$ so that it is compatible at the boundary, inductively. Smoothing differential forms are performed by using a continuous family of multisections. See [28] Section 7.5 and [25, 26].

An advantage of this method is that it is easier to keep symmetry. On the other hand, the author does not know how to work over $\mathbb{Q}$ coefficient when we use de Rham cohomology.

The other method is to use the notion of Kuranishi homology proposed by [42]. Namely we regard a pair $(\mathcal{M}, f)$ of the space with Kuranishi structure $\mathcal{M}$ and a strongly continuous (weakly submersive) map $f : \mathcal{M} \to Y$ as a chain on Y Joyce called it a Kuranishi chain and construct the structure on the chain complex consisting of Kuranishi chains. We can define the fiber product among Kuranishi

chains always. So using this method we can construct a structure on the chain complex of Kuranishi chains without perturbing the equation at all[21]. A difficulty of this construction is as follows:

When we define a homology theory by regarding 'a pair of a space and a map from it' as a chain, we need to eliminate the automorphism by some method. Otherwise it does not give a correct homology group.

Let us consider the pair of a manifold with corner N and a map $f : N \to Y$. We identify $(N, f) \sim (N', f \circ h)$ for each diffeomorphism $h : N' \to N$. We take the set of equivalence classes of this identifications and let $SM(Y)$ be the free abelian group whose basis is identified with this set. We define a boundary operator $\partial(N, f) = (\partial N, f)$. We thus obtain a chain complex. However its homology is not isomorphic to the ordinary homology of Y. (The reason is, roughly speaking, the diffeomorphism provides too big freedom of identifications.

Joyce resolved this problem by including 'gauge fixing data' as a part of the data of Kuranishi chain and eliminate the automorphism.

The construction of this section is not canonical and many choices are involved during the construction. However we can show that the resulting A_∞ algebra on the cohomology group $H(L; \Lambda_0)$ is independent of those choices up to an isomorphisms of filtered A_∞ algebra. To prove it we construct an appropriate A_∞ homomorphism between the A_∞ structures on singular chain complex and show that it is a homotopy equivalence. We do so by inductively constructing chain maps using an appropriate moduli spaces.

Remark 6.3. To prove the well-defined-ness of the structure up to homotopy equivalence, we first need to build a homotopy theory of filtered A_∞ algebra. There are various references ([52, 74, 45]) describing homotopy theory of A_∞ algebra (without filtration). However, for example, it is hard to find a reference where the equivalence of various definitions of homotopy between A_∞ homomorphisms is proved in detail. (The author knows at least two different definitions.) See [28] Chapters 4 and 5 for the homotopy theory of filtered A_∞ algebra. Homotopy theory of A_∞ algebra is regarded as a generalization of the homotopy theory of differential graded algebra ([77]).

Once we obtain a structure of filtered A_∞ algebra on the singular chain complex then we can use homological algebra to squeeze it to the homology group. (This is a classical result which goes back to Kadeišvili [43]. See also [49].) We thus obtain a structure of filtered A_∞ algebra on the cohomology group.

Remark 6.4. We omit several important parts of the proof. Especially we omit the argument on the orientation and sign. The (relative) spin structure is used to orient the moduli spaces $\mathcal{M}(L; \beta)$. We remark that proving the orientability of $\mathcal{M}(L; \beta)$ is only the first (and rather easier) step of the whole works on sign and orientation. To work out the sign and orientation part of the construction of the A_∞ structure, we need the following: Choose orientations of many spaces so that their fiber products are related at the corners and the boundaries. To check those orientations are consistent to the sign appearing in the homological algebra of A_∞ structures. This is heavy and cumbersome job, and is performed in Chapter 8 [28], which occupies around 80 pages.

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[21]Perturbation becomes necessary to study the relation between Kuranishi homology with other homology theories such as singular homology.

7. A_∞ CATEGORY AND HOMOLOGICAL MIRROR SYMMETRY

There are two kinds of applications of Lagrangian Floer theory. One is to the symplectic geometry and the other is to the mirror symmery. Application to the symplectic geometry is described for example in [61]. So we focus here to the application to the mirror symmetry. A important problem where Lagrangian Floer theory is related to mirror symmetry is M. Kontsevitch's homological mirror symmetry conjecture [46]. We discuss it in this article.

Definition 7.1. A *filtered A_∞ category* $\mathcal{C}$ consists of the set of objects $\mathfrak{Ob}(\mathcal{C})$, the set of morphisms $\mathcal{C}(c, c')$ for each $c, c' \in \mathfrak{Ob}(\mathcal{C})$, such that $\mathcal{C}(c, c')$ is a Λ_0 module, and a Λ_0 module homomorphisms

$$\mathfrak{m}_k : \bigotimes_{i=1}^{k} \mathcal{C}(c_{i-1}, c_i) \to \mathcal{C}(c_0, c_k)$$

for each $c_0, \cdots, c_k \in \mathfrak{Ob}(\mathcal{C})$, $k = 1, \cdots, \infty$. We assume the relation (4.1) among them.

Example 7.2. In case $\mathfrak{m}_k = 0$ for $k \neq 2$, the A_∞ category becomes a usual additive category. Note $\mathfrak{m}_2$ is different from the composition by sign.

The case $\mathfrak{m}_k = 0$ for $k \neq 1, 2$ is called the *differential graded category*. It is introduced by [9].

To each symplectic manifold (X, ω), we can associate an filtered A_∞ category $\mathcal{LAG}(X)$ whose object is a pair (L, b) of a spin Lagrangian submanifold L and $b \in \mathcal{M}(L)$. In case of $c_i = (L, b_i)$, namely in the case when Lagrangian submanifolds L are the same, we define $\mathcal{LAG}(X)((L, b_i), (L, b_{i+1})) \cong H(L; \Lambda_0)$ and

$$\mathfrak{m}_k(x_1, \cdots, x_k) = \sum_{\ell_0=0}^{\infty} \cdots \sum_{\ell_k=0}^{\infty} \mathfrak{m}_{\ell_0+\cdots+\ell_k+k}(b_0^{\otimes \ell_0}, x_1, b_1^{\otimes \ell_1}, \cdots, b_{k-1}^{\otimes \ell_{k-1}}, x_k, b_k^{\otimes \ell_k}).$$

Here the right hand side is $\mathfrak{m}$ in Theorem 4.1, and the left hand side is $\mathfrak{m}$ in Definition 7.1. The detail of the definition of this A_∞ category is in [20].

A_∞ category is not an abelian category. However we can replace the notion of chain complex in abelian category by the twisted complex and can define its derived category [46].

Here the twister complex is defined as follows. Let $c_i \in \mathfrak{Ob}(\mathcal{C})$, $i = 1, \cdots, n$, $x_{ij} \in \mathcal{C}(c_i, c_j)$, $1 \leq i \leq j \leq n$. We say $(\{c_i\}, \{x_{ij}\})$ is a *twisted complex* if, for each $1 \leq a, b \leq k$, the formula

$$(7.1) \qquad \sum_{k} \sum_{j_0=a, \cdots, j_k=b} \mathfrak{m}_k(x_{j_0 j_1}, \cdots, x_{j_{k-1} j_k}) = 0$$

is satisfied.

In case $\mathfrak{m}_k = 0$ for $k \neq 2$ and $x_{ij} = 0$ for $j \neq i+1$, (7.1) becomes the relation $\mathfrak{m}_2(x_{i+1 i+2}, x_{i i+1}) = 0$. Namely it defines a chain complex.

We can define a mapping cone of a morphism of twisted complex. We thus obtain a triangulated category. See [20, 72]. We call this triangulated category the *derived category* of $\mathcal{C}$.

The homological mirror symmetry conjecture by Kontsevitch [46] is as follows.

Conjecture 7.3. For each Calabi-Yau manifold X we can associate another Calabi-Yau manifold $\hat{X}$, its mirror, such that the derived category of the category of coherent analytic sheaves on $\hat{X}$ is equivalent to the derived category of $\mathcal{LAG}(X)$.

Remark 7.4. A Calabi-Yau manifold is a Kähler manifold and so has both symplectic structure (Kähler form) and the complex structure. In Conjecture 7.3 we consider the symplectic structure of X, and the complex structure of $\hat{X}$, only. The former is called the *A-model* and the later is called the *B-model* [81].

The statement of the above conjecture is slightly imprecise. Let us explain this point first. The problem is the following: In the category $\mathcal{LAG}(X)$ the set of morphisms is a module over the universal Novikov ring Λ_0. On the other hand, the set of morphisms in the category of coherent analytic sheaves of X is a complex vector space. So they can not be isomorphic. There are two ways to correct this point.

(1) We substitute a sufficiently small positive number for T. Then the set of morphisms of $\mathcal{LAG}(X)$ becomes a $\mathbb{C}$ vector space. For this purpose we need to show that the structure constants of $\mathfrak{m}_k$ converges when we substitute a sufficiently small positive number for T. This is actually very difficult to prove.

(2) We modify the category of coherent analytic sheaves on $\hat{X}$ so that the set of morphisms becomes a module over the Novikov ring [49, 23].

To realize the plan (2), we regard $\hat{X}$ not as a single Calabi-Yau manifold but a family of it parametrized by a disc $D^2(\epsilon)$ with small diameter. Namely we consider a family $\pi : \mathfrak{X} \to D^2(\epsilon)$.

We assume all the fibers of π other than $\pi^{-1}(0)$ is nonsingular. The projection π is not arbitrary. In mirror symmetry the case when 0 is a maximal degenerate point appears. Here 0 is a said to be a maximal degenerate point, if $\pi^{-1}(0)$ is a union of irreducible components which are normal crossing and that there exists a point in $\pi^{-1}(0)$ where $\dim_{\mathbb{C}} X + 1$ irreducible components meet. For example, let us define a family of degree $n + 1$ hyper surfaces of $\mathbb{C}P^n$ by

$$\{([x_0 : \cdots : x_n], t) \in \mathbb{C}P^n \times D^2 \mid x_0 \cdots x_n = t(x_0^{n+1} + \cdots + x_n^{n+1})\}.$$

Then the fiber of $\pi : ([x_0 : \cdots : x_n], t) \mapsto t$ at t is nonsingular for $t \neq 0$ and is singular at $t = 0$. Moreover the n irreducible components meet at a point in the fiber of $t = 0$.

Now we suppose that 0 is a maximal degenerate point. We formalize the family along the fiber of 0. We then obtain a formal scheme over $\mathbb{C}[[t]]$. The set of the morphisms of the category of its coherent sheaves is a module over $\mathbb{C}[[t]]$. When we include the branched covering whose branch locus is in 0, then the coefficient ring becomes the Puiseux series ring, that is very close to the Novikov ring. Later on in various part of the story we may either work on Novikov ring or on $\mathbb{C}$. We need some nontrivial arguments to go from one to the other. We however omit the argument to do so[22].

We can state a part of the homological mirror symmetry more explicitly as follows.

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[22]Actually we do not know the way to go from one to the other completely.

Conjecture 7.5. ([22]) For each pair (L, b) of Lagrangian submanifold L of X and $b \in \mathcal{M}(L)$, we can associate a chain complex $\mathcal{E}(L, b)$ of the coherent analytic sheaves over $\hat{X}$, such that the following holds.

(1) There exists an isomorphism[23]

$$HF((L_1, b_1), (L_2, b_2)) \cong \operatorname{Ext}(\mathcal{E}(L_1, b_1), \mathcal{E}(L_2, b_2)).$$

(2) The following diagram commutes.

$$HF((L_1, b_1), (L_2, b_2)) \otimes HF((L_2, b_2), (L_3, b_3)) \xrightarrow{\mathfrak{m}_2} HF((L_1, b_1), (L_3, b_3))$$

$$\cong \Big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong \Big\downarrow$$

$$\operatorname{Ext}(\mathcal{E}(L_1, b_1), \mathcal{E}(L_2, b_2)) \otimes \operatorname{Ext}(\mathcal{E}(L_2, b_2), \mathcal{E}(L_3, b_3)) \longrightarrow \operatorname{Ext}(\mathcal{E}(L_1, b_1), \mathcal{E}(L_3, b_3))$$

Here the lower horizontal arrow is the Yoneda product and the vertical arrows are the isomorphisms in (1)[24].

8. HOMOLOGICAL MIRROR SYMMETRY AND CLASSICAL MIRROR SYMMETRY

The classical mirror symmetry is a statement which claims the coincidence of the generating function of the number of pseudo-holomorphic curves on X and the generating function obtained from the deformation theory of complex structures (Yukawa coupring) of $\hat{X}$. In this section, we explain its relation to homological mirror symmetry.

For each pair of A_∞ categories $\mathcal{C}_1, \mathcal{C}_2$, there exists an A_∞ category $\mathcal{FUNC}(\mathcal{C}_1, \mathcal{C}_2)$ such that its objects is an A_∞ functor $\mathcal{C}_1 \to \mathcal{C}_2$ [20].

The identity functor $\mathcal{C} \to \mathcal{C}$ is an object of $\mathcal{FUNC}(\mathcal{C}, \mathcal{C})$, which we denote by $1_\mathcal{C}$. The set of (pre) natural transformations from $1_\mathcal{C}$ to $1_\mathcal{C}$ becomes an A_∞ algebra [20]. We denote it by $\mathfrak{Hom}(1_\mathcal{C}, 1_\mathcal{C})$. If $\mathcal{C}$ is an A_∞ category with one object only, that is nothing but an A_∞ algebra C, then $\mathfrak{Hom}(1_\mathcal{C}, 1_\mathcal{C})$ coincides with the Hochschild complex $CH(C, C) = (\bigoplus_k Hom(C^{\otimes k}, C), \delta)$. (In case C is an associative algebra it coincides with the Hochschild complex in the usual sense.) In the case of general A_∞ category $\mathcal{C}$, we call $\mathfrak{Hom}(1_\mathcal{C}, 1_\mathcal{C})$ the *Hochschild complex* also.

Conjecture 8.1. Under certain assumption on X[25], the Hochschild cohomology $H(\mathfrak{Hom}(1_\mathcal{C}, 1_\mathcal{C}))$ of $\mathcal{C} = \mathcal{LAG}(X)$ is isomorphic to the quantum cohomology ring $QH(X; \Lambda)$ of X.

Conjecture 8.1 claims that $\mathcal{LAG}(X)$ determines the quantum cohomology ring.

On the other hand, we can define a similar Hochschild complex from the derived category of the category of coherent analytic sheaves on $\hat{X}$. The cohomology of this Hochschild complex and its product structure determines the deformation theory of $\hat{X}$ and Yukawa coupling on the deformation space of it. Thus Conjecture 8.1 implies the equality

$$\text{quantum cup product} = \text{Yukawa coupling},$$

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[23]Ext is the derived functor of the functor which associate $Hom(\mathcal{E}_1, \mathcal{E}_2)$ to a pair of the coherent analytic sheaves $\mathcal{E}_1, \mathcal{E}_2$.

[24]In Theorem 4.1 $\mathfrak{m}_2$ is defined on the singular cohomology $H(L(u); \Lambda_0)$. Since $\mathfrak{m}_2$ is a derivation with respect to the boundary operator $\mathfrak{m}_1$, it follows that $\mathfrak{m}_2$ defines a product structure on the $\mathfrak{m}_1$ cohomology, that is the Floer cohomology.

[25]For example the author believes that all the projective Calabi-Yau manifolds have this property.

that is the classical mirror symmetry[26]. The discussion above seems to be written in [46]. Seidel [70] states it as Conjecture 4 including the case when X is not necessarily compact.

Let us discuss Conjecture 8.1 more. The map

$$(8.1) \qquad QH(X; \Lambda_0) \to H(\mathfrak{Hom}(1_\mathcal{C}, 1_\mathcal{C}))$$

can be defined by Open-Closed Gromov-Witten theory (See [28] section 3.8.) as follows. Let us consider the set $\widetilde{\mathcal{M}}^{\mathrm{reg}}_{k+1}(L; \beta)$ we defined in section 4. We divide it by the action of $U(1) \cong \{g \in \mathrm{Aut}(D^2, j_{D^2}) \cong PSL(2; \mathbb{R}) \mid g(0) = 0\}$ and compactify it. We denote it by $\mathcal{M}_{1,k+1}(L; \beta)$. We can define $ev : \mathcal{M}_{1,k+1}(L; \beta) \to L^{k+1}$. Moreover we define

$$ev^{\mathrm{int}} : \mathcal{M}_{1,k+1}(L; \beta) \to M$$

by

$$ev^{\mathrm{int}}(u) = u(0).$$

If Q is a cycle of M then we can use fiber product to define:

$$\mathcal{M}_{1,k+1}(L; \beta; Q) = \mathcal{M}_{1,k+1}(L; \beta) \; _{ev^{\mathrm{int}}} \times_M Q.$$

This space has a Kuranishi structure. We can use it in place of $\mathcal{M}_{k+1}(L; \beta)$ and proceed in the same way as section 6. We then obtain:

$$\mathfrak{q}(Q; \cdots) : H(L; \Lambda_0)[1]^{\otimes k} \to H(L; \Lambda_0)[1].$$

Namely we obtain an element of $CH(HF((L, b), (L, b)), HF((L, b), (L, b)))$ for each of L, b. They behave in a functorial way when we move L, b. It thus defines an element of $H(\mathfrak{Hom}(1_\mathcal{C}, 1_\mathcal{C}))$. We associate it to $[Q] \in H(M)$ and obtain the map (8.1). We remark that $QH(X; \Lambda_0)$ is isomorphic to the sigular cohomology as a module over Λ_0. (The ring structure is deformed.) We can perform the above construction and prove that (8.1) is a ring homomorphism without assuming any extra condition on X. The proof of later is similar to the proof of the associativity of the quntum cup product. Seidel proved it in the case Lagrangian submanifold is exact[27]. In [6][28] the case when L is monotone is proved.

The hardest part of the proof of the Conjecture 8.1 is the proof that (8.1) is an isomorphism. This does not hold unless we assume some condition to X. In fact, there is a symplectic torus which does not contain a Lagrangian submanifold without nontrivial Floer cohomology. For such X, the homomorphism (8.1) is not an isomorphism. Proving that (8.1) is an isomorphism is proving the existence of enough many Lagrangian submanifolds so that they distinguish all the cohomology classes of X. It may be regarded as a 'mirror to the Hodge conjecture' and is a very difficult problem to solve in general. There are various cases where the map (8.1) is proved to be an isomorphism. It is proved that (8.1) is an isomorphism in the case of toric manifold in [32].

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[26] We did not give a precise description how the quantum cohomology is related to the deformation theory of its mirror, but only suggests that there should be some relation. The author does not know more precise way to state this relation. The author does not know a reference where such relation is established or at least conjectured in a precise way either.

[27]Namely there exists one form θ on M such that $\omega = d\theta$ for the symplectic form ω, and there exists a function f on L such that $\theta = df$ on L.

[28]Note Floer cohomology is called the Lagrangian quantum homology in [6]. I have a strong objection to the authors of [6] who try to chang the name of the notion which had been established long time ago and try to eliminate the name of Floer who discovered this important notion.

The homological algebra of the map (8.1) is closely related to a conjecture by Deligne which is solved in [48].

9. STROMINGER-YAU-ZASLOW CONJECTURE

We next describe a relation between homological mirror symmetry conjecture and a conjecture by Strominger-Yau-Zaslow [76]. Let us consider a way to construct $\hat{X}$ so that the homological mirror symmetry[29] holds. For each point p of a complex manifold $\hat{X}$, we can define a skyscraper sheaf $\mathcal{F}_p$. Namely we put $\mathcal{F}_p(U) = \mathbb{C}$ if $p \in U$ and $\mathcal{F}_p(U) = 0$ if $p \notin U$. Let us assume that there exists (L, b) such that $\mathcal{E}(L, b) = \mathcal{F}_p$ by the correspondence in Conjecture 7.5. We put $(L_p, b_p) = (L, b)$ Then Conjecture 7.5 (1) implies

$$HF((L_p, b_p), (L_p, b_p)) \cong \text{Ext}(\mathcal{F}_p, \mathcal{F}_p).$$

We can calculate the Ext group of the skyscraper sheaf easily and can show that the right hand side is isomorphic to $H(T^n; \mathbb{C})$ that is the cohomology of the n-dimensional torus. Floer cohomology $HF((L_p, b_p), (L_p, b_p))$ is related to the cohomology of L_p by the spectral sequence in Theorem 2.1 (2). In case the spectral sequence degenerates we have $H(L_p) \cong H(T^n)$. Thus we conjecture that the Lagrangian submanifold L_p is a torus.

We next consider b_p. In general it is an odd degree cohomology class of L_p. When we study Calabi-Yau manifold, the Floer cohomology is not only $\mathbb{Z}_2$ graded but is $\mathbb{Z}$ graded (See [68]). We need to assume $b_p \in H^1(L_p)$ for $\mathbb{Z}$ grading. In this case b_p corresponds to an element of $Hom(H_1(L_p; \mathbb{Z})), \mathbb{C} \setminus \{0\})$ that is a flat connection. It is believed that this flat connection is unital[30]. In sum we have:

The mirror manifold $\hat{X}$ is a moduli space of the pair of Lagrangian torus L_p and a flat $U(1)$ connection on it.

Lagrangian torus in a symplectic manifold appears in the study of integrable system. Namely if (X, ω) is a symplectic manifold and $\pi : X \to B$ is a map whose fiber is a compact Lagrangian submanifold, then we can show that the fiber of π is a torus[31].

For each $q \in B$ we put $L_q = \pi^{-1}(q)$. The moduli space of flat $U(1)$ connections on L_q becomes the dual torus of L_q. In other words, the mirror $\hat{X}$ of X is a union of the dual tori of the fibers.

We may sum up the above discussion to the following conjecture by Strominger-Yas-Zaslow. We need to include the case when there is a singular fiber. We consider a map $\pi : X \to B$ from a $2n$-dimensional symplectic manifold (X, ω) to an n-dimensional manifold B, so that the fiber of the generic point is an n-dimensional Lagrangian torus. Let B_0 be the subset of B consisting of the points whose fiber is an n-dimensional Lagrangian submanifold.

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[29]From the point of view of [76], it may be better to say D-brane duality.

[30]When we work over $\mathbb{C}$ coefficient (and not over Novikov ring coefficient) the parameter which deforms the connection to non-unitary is $Hom(H_1(L_p; \mathbb{Z}), \mathbb{R}_{>0})$ and becomes a part of the parameter to deform the Lagrangian submanifold L.

[31] This follows from Liouville-Arnold theorem which asserts that if there are n-independent first integral and if the level set of them is compact, then the level set is a torus. The coordinate of the fiber is called the angle coordinate and the coordinate of B is called the action coordinate.

Conjecture 9.1. For each maximal degenerate family of Calabi-Yau manifolds, we have a projection $\pi : X \to B$ such that $B \setminus B_0$ is of codimension ≥ 2 in B. We obtain a family $\hat{X}_0 \to B_0$ by taking fiber-wise dual. $\hat{X}$ is a compactification of $\hat{X}_0$.

Remark 9.2. Strominger-Yas-Zaslow conjectures the fibers L_q to be special Lagrangian submanifolds[32]. Nowadays it is known that we need some modification on this points. See [41, 14].

In the case when Conjecture 9.1 holds, we expect to obtain the assignment $(L, b) \mapsto \mathcal{E}(L, b)$ in Conjecture 7.5 as follows. For simplicity we assume that L is transversal to the fibers of $\pi : X \to B$. A point p of $\hat{X}$ is identified with $(L_{q(p)}, b(p))$ by mirror symmetry. Here $q(p) \in B$ and $L_{q(p)}$ is its fiber. The element $b(p)$ determines a flat bundle on it and we may regard it as an element of $\mathcal{M}(L_{q(p)})$.

Conjecture 9.3. ([21, 24]) The coherent sheaf $\mathcal{E}(L, b)$ is obtained by the holomorphic bundle on $\hat{X}$ whose fiber at p is identified with the Floer cohomology $HF((L_{q(p)}, b(p)), (L, b))$.

Conjecture 9.3 provides a way to construct $\mathcal{E}(L, b)$, as follows. We consider the family of Floer cohomologies $HF((L_{q(p)}, b(p)), (L, b))$ where $p \in \hat{X}$ moves. If we can define a holomorphic structure on it so that it becomes a holomorphic vector bundle on $\hat{X}$, then we can *define* $\mathcal{E}(L, b)$ to be this holomorphic vector bundle.

This idea was discovered during the authors' discussion with M. Kontsevitch at the year 1998 during his stay in IHES. It was realized in the case of Abelian variety or complex torus in [22]. Note in those cases, there are no singular fiber. So the situation is simpler. We remark that in the case of elliptic curve, where we can calculate the both sides of the (homological) mirror symmetry conjecture directly, homological mirror symmetry was studied earlier by [46, 64]. We also refer [49, 2] for homological mirror symmetry of complex torus.

Let us discuss the case of general Calabi-Yau manifold where there is a sigular fiber. We describe the way how we obtain a complex structure on $\hat{X}_0$ in Conjecture 9.1. We expect to perform the construction in two steps. In the first step, we use flat affine structure on B and local tensor calculus to define a complex structure. (This complex structure is called *semi-flat*.) In the second step, we add the corrections that are induced by the pseudo-holomorphic discs[33]. This correction is called the *instanton correction.*

The semi-flat complex structure is defined as follows. We consider $\pi : X \to B$. For $q \in B_0$ we have a canonical isomorphism $H^1(L_q; \mathbb{R}) \cong T_q B$. (Here $L_q = \pi^{-1}(q)$.) We use the lattice $H^1(L_q; \mathbb{Z})$ of $H^1(L_q; \mathbb{R})$ to define a flat affine structure on B.

On the other hand, the moduli space of flat unitary connections of the fibers is locally the cohomology group $H^1(L_q; \sqrt{-1}\mathbb{R})$ with coefficient in the Lie algebra $\sqrt{-1}\mathbb{R}$ of $U(1)$. Conjecture 9.1 asserts that these two determine $\hat{X}$ locally. Therefore the tangent space $T_q \hat{X}$ is identified with $H^1(L_q; \mathbb{R}) \oplus H^1(L_q; \sqrt{-1}\mathbb{R})$, that has a complex structure induced by J_0 [34].

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[32]Especially it is conjectured to be a minimal submanifold.

[33]There is no correction in the case of complex torus.

[34]There are various reference on the construction of this complex structure J_0. The author quote as many reference as he knows on this point in section 2 of [24]. So we omit those references here.

We next describe the instanton correction. The existence of instanton correction is related to the singular fiber as follows. The semi-flat complex structure J_0 can be constructed only on $\hat{X}_0$. Its compactification $\hat{X}$ contains a (dual to) the singular fibers. The complex structure J_0 however does not extend to $\hat{X}$. Therefore we need a correction to extend it to a complex structure on $\hat{X}$. Existence of such an instanton correction or quantum correction had been know in the physics literature. (See for example [62].) This phenomenon is related to the wall crossing of the Floer cohomology as was observed in [22, 27]. One method to study it is to reduce it to Morse homotopy[35], that is equivalent to the enumeration of pseudo-holomorphic curves via tropical geometry. Relation to Morse homotopy is discussed in [34, 49, 24]. The case of cotangent bundle is discussed in [34]. In [49] it is claimed that one can generalize [34] to the case when the fiber is a torus. The study of singular fiber then is started in [24]. See [56] for tropical geometry.

In [24] the author discussed the way how the instanton correction is related to the Gromov-Witten invariant. There Dolbeault cohomology is used to describe deformation of complex structure. In [24], the deformation of $\overline{\partial}$ operator is described by the singular current that has a support on the wall (of the wall crossing of Lagrangian Floer cohomology). The wall becomes more and more dense when we consider the deformation of higher and higher degree. Also in [24] the Feynman diagram of 0-loop is used to describe the scattering of the walls.

After [24] had been submitted for publication, Kontsevich-Soibelman [50] discussed the same phenomenon using Čech cohomology and studied the deformation of the complex structure as the deformation of the coordinate change instead of the deformation of $\overline{\partial}$ operator. They introduced a nilpotent group of coordinate changes which has a filtration by the order of the deformations. (This group coincides with the group of A_∞ automorphisms of $H(T^n; \Lambda_0)$ which preserves the volume element.) [50] study the case of K3 surface mainly.

After these works had been done, those pictures were used by Gross-Siebert, who had been working on a similar problem independently (in [37] for example). They proved a reconstruction theorem from toric degeneration in general dimension in [38].

Thus the program (due to [24] etc.) to show the mirror symmetry which asserts a relation between 'symplectic geoetry' and 'complex geometry' through 'Morse homotopy' (or equivalently through 'tropical geometry') is mostly realized for the part 'Morse homotopy' $\Rightarrow$ 'complex geometry', (as far as the part where Lagrangian submanfolds or coherent sheaves are not included). The study of the other part of the program is making progress[36].

We discussed in this section the study of mirror symmetry based on the family Floer cohomology or Strominger-Yau-Zaslow conjecture. One of the other important study of homological mirror symmetry of Calabi-Yau manifold is its proof in the case of quartic surface [71] by P. Seidel. This is based on Seidel's study of directed A_∞ category associated to the symplectic Lefschetz pencil. We omit it and refer [71]. (Note in [71] there is a warning that the proof of a part of the results which is used in [71] is not written up in detail. However now all the necessary results are established by [72]. Therefore the proof of the main results of [71] is

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[35]In other words, we use the coincidence of the moduli space of pseudo-holomorphic discs in the cotangent bundle and the set of the solution of an ordinary equation to a map from a graph.

[36]For example the case of cotangent bundle is discussed in [35, 57].

now completed.) This theory of Seidel is also important in the study of mirror symmetry in the non-Calabi-Yau case, which we discuss in the next section.

10. Mirror symmetry in the non-Calabi-Yau case

So far we discussed mirror symmetry for Calabi-Yau manifold only. In this last section, we discuss non-Calabi-Yau case. The mirror of a manifold X is expected to be another manifold $\hat{X}$ in the Calabi-Yau case. However in a more general situation a mirror to a manifold X becomes a pair $(\hat{X}, W)$ of a manifold $\hat{X}$ and a function W on it. The function W is called the *Landau-Ginzburg super potential*. We explain this point here.

We first consider the case when X is a symplectic manifold (X, ω). As we explained in the last section, the mirror $\hat{X}$ of a Calabi-Yau manifold X is regarded as the moduli space of skyscraper sheaves. The homological mirror symmetry conjectures that the moduli space of skyscraper sheaves is identified with a moduli space of the pair (L, b) where L is a Larangian submanifold of X and b is a flat unitary connection on it ($b \in H^1(L; \sqrt{-1}\mathbb{R})$). Using it to construct $\hat{X}$ is an ideal of the construction of $\hat{X}$.

In the case of Calabi-Yau manifold the space $\mathcal{M}_{\text{weak}}(L)$ (that is defined in section 2) coincides with $\mathcal{M}(L)$. In other words, the potential function $\mathfrak{PO}$ is a constant function 0[37].

In geneural, the function $\mathfrak{PO}$ may not be 0. We regard the mirror to (X, ω) as the pair $(\hat{X}, W)$, where $\hat{X}$ is (an irreducible component of) the moduli space of the pair (L, b) where L is a Lagrangian submanifold and $b \in \mathcal{M}_{\text{weak}}(L)$ and W is the function

$$(L, b) \mapsto W(L, b) = \mathfrak{PO}(b).$$

A typical case where such a construction works is the case when X is Fano or toric. Below we discuss on the toric case. ([3] is a good reference of the material in this section. In [44] some applications to the algebraic geometry are also discussed.) Toric manifold X has a T^n action so that its non-degenerate orbits are Lagrangian submanifolds. The orbits are the fibers of the moment map $X \to P$. The fiber $L(u)$ of the interior point u of P that is a polytope in $\mathbb{R}^n$ is diffeomorphic to T^n. As we mentioned in section 3, we have $H^1(L(u); \Lambda_0) \subset \mathcal{M}_{\text{weak}}(L)$.

If we take an element b of $H^1(L(u); \sqrt{-1}\mathbb{R})$ as bounding cochain, then b corresponds to a flat unitary connection on $L(u)$. The totality of such $(L(u), b)$ where u is a interior point of P, is the union of the dual torus of fibers $L(u)$. We regard $\hat{X}$ as such totality[38].

We consider the function $W = \mathfrak{PO}$ on $\hat{X}$. Then Theorem 3.1 claims that the Floer cohomology of (L, b) is nonzero if and only if W is critical at (L, b).

On the other hand, as we mentioned right before Theorem 2.2, the Floer cohomology $HF((L_1, b_1), (L_2, b_2))$ is defined only if $\mathfrak{PO}(b_1) = \mathfrak{PO}(b_2)$.

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[37]We can prove it by the dimension counting argument. In the case of Calabi-Yau manifold, we consider L with vanishing Maslov index. So $\mathfrak{m}_k(b, \cdots, b)$ never have a degree of the fundamental class $[L]$, if $b \in H^1(L)$.

[38]In this section we omit the argument which is related to the choice of the coefficient ring. Namely the difference of Λ_0 coefficient and $\mathbb{C}$ coefficient. I think the universal Novikov ring Λ_0 is actually the correct choice. In that case $\hat{X}$ becomes a rigid analytic space.

Conjecture 10.1. $\mathcal{LAG}(X)$ splits into the direct sum of the filtered A_∞ categories associated to each of the critical values of W. The critical value of W coincides with the eigenvalue of the linear map $H(X; \Lambda_0) \to H(X; \Lambda_0)$ that is given by $x \mapsto x \cup^Q c_1(X)$, where $c_1(X)$ is the first Chern class and $\cup^Q$ is the quantum cup product.

The author heard this conjecture for the first time in a talk by M. Kontsevich at Vienna (2006). (He was also informed that a similar conjecture was made by several people independently.)

In the toric case, the nontrivial part of Conjecture 10.1 is stated as follows. Let us consider L and $b \in \mathcal{M}_{\text{weak}}(L)$ which is not necessary a T^n orbit. Suppose $HF((L, b), (L, b)) \neq 0$. Then Conjecture 10.1 states that $\mathfrak{PO}(b)$ is equal to a critical value of $W = \mathfrak{PO} : \hat{X} \to \Lambda_0$. (Note W is defined on the set of $(L(u), b)$ where $L(u)$ is a T^n orbit and $b \in H^1(L(u); \Lambda_0)$.)

We hope to prove this statement by using the fact that T^n orbits generates the category $\mathcal{LAG}(X)$ in certain sense. The proof of the second half is closely related to the map (8.1).

Suppose that c is a critical point of W. Then $W^{-1}(c) = \hat{X}_c$ has a singularity.

Conjecture 10.2. The derived category of the direct sum factor $\mathcal{LAG}(X)$ that corresponds to the critical value c is isomorphic to the derived category of the category of the matrix factorization of a singularity of $\hat{X}_c$.

We refer [80] for the matrix factorization.

Remark 10.3. The conjectures in this section seem to be easier to prove than the conjectures in the last section. We hope to solve them in a near future. One of the reasons whey they are easier is that in the toric case the complex structure of $\hat{X}$ has no instanton correction. In case when X is not toric but is Fano, the complex structure of $\hat{X}$ may have instanton correction. However the examples in [3] suggest that the instanton correction is simpler in Fano case than Calabi-Yau case, and easier to study.

The conjectures 10.1 are 10.2 are a version of homological mirror symmetry. The conjecture which is a version of classical mirror symmetry can be stated as follows. The quantum cohomology of X (or the Frobenius structure induced by the big quantum cohomology by [13]) coincides with Saito's flat structure [66] associated to W. (See also [67].) This statement has been discussed in Givental [36] or Hori-Vafa [40] and has been applied successfully. The fact that W is the potential function of [28] was conjectured by them also. This fact is established by [11, 30, 31].

So far we put superpotential W in the complex sides. In fact, we defined the function W by using the Floer theory of Lagrangian submanifold on its mirror. When we consider W in the symplectic side, then its mirror is a complex manifold. So it is not natural to study Floer theory on the mirror. This is the reason why I discussed the case when W is on the complex sides, so far.

In fact, however, more results have been obtained already in the case W is in the symplectic side.

To the pair (X, W) of symplectic manifold and a function on it, Seidel associated a directed A_∞ category $\mathcal{LAG}(X, W)$, under certain exactness condition[39]. (Seidel mentioned that his construction is suggested by M. Kontsevich's talk.) The object of the Seidel's directed A_∞ category is a Lagrangian submanifold that is a vanishing cycle of W. The morphism space is a version of Floer cohomology. They are defined in detail in [72].

The mirror of such a pair (X, W) is conjectured to be a compact Fano manifold $\hat{X}$, in many cases. Namely:

Conjecture 10.4. The derived category of directed A_∞ category $\mathcal{LAG}(X, W)$ associated to (X, W) is equivalent to the derived category of the category of coherent sheaves of $\hat{X}$.

It is known that for many of the Fano manifolds, the derived category of the category of its coherent sheaves has a distinguished generator[40]. It is conjectured that those distinguished generator becomes the vanishing cycle of W by the above mentioned isomorphism. This conjecture is checked by many people. (See [69, 4, 79, 5, 1] for example.)

The formulation of the mirror symmetry for non-Calabi-Yau case is not yet completed.

The Japanese version of this article was written in 2009. The reference below so is restricted to those which the author already knew at that time.

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[39]We say M is exact if there exists a differential one form θ such that $\omega = d\theta$. We say a Lagrangian submanifold L in it is exact if there exists a function f on L such that $\theta = df$ holds on L.

[40] Such generator is given in the case of $\mathbb{C}P^n$ by [8].

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